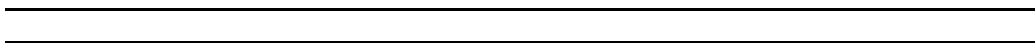


# Coarse embedding into uniformly convex Banach space

Qinggang Ren<sup>1</sup>

*Department of Mathematics, Kyoto University, Kyoto, Japan, 606-8502.*



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*Email address:* `qinggang.ren@hw4.ecs.kyoto-u.ac.jp` (Qinggang Ren)

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## Abstract

In this paper, we study the coarse embedding into Banach space. We proved that under certain conditions, the property of embedding into Banach space can be preserved under taking the union the metric spaces. For a group  $G$  strongly relative hyperbolic to a subgroup  $H$ , we proved that if  $H$  admits a coarse embedding into a uniformly convex Banach space, so is  $B(n) = \{g \in G \mid |g|_{S \cup \mathcal{H}} \leq n\}$ .

*Keywords:* Coarse embedding, Uniformly Banach space, Relative hyperbolic group

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## 1. Introduction

After Gromov pointed out that the coarse embedding(also refer as uniform embedding) should be relevant to Novikov conjecture[7][6], G.Yu introduced a property called property A for discrete metric spaces[14]. A metric space with property A admits a coarse embedding into a Hilbert space. And G.Yu proved the coarse Baum-Connes conjecture holds for the metric spaces with bounded geometry, which admits a coarse embedding into a Hilbert space[14]. Subsequently G.Kasparov and G.Yu proved the coarse geometric Novikov conjecture holds for discrete metric spaces with bounded geometry, which admits a coarse embedding into a uniformly convex Banach space[10]. Coarse embedding into Hilbert space has been studied deeply these years, see [3]. But there are less results on the coarse embedding into uniformly convex Banach space. Also, V.Lafforgue constructed an example which can not be coarse embedded into uniformly convex Banach spaces[11]. We should

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*Email address:* `qinggang.ren@hw4.ecs.kyoto-u.ac.jp` (Qinggang Ren)

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mention that N. Brown and E. Guentner proved that every metric space with bounded geometry admits a coarse embedding into a strictly convex and reflexive Banach space[2].

In this paper, we study the coarse embedding into a uniformly convex Banach space. We first rewrite the condition for coarse embedding into a uniformly Banach space.

**Theorem 1.1.** *Let  $X$  be a metric space and  $E$  be a Banach space, and  $1 \leq p < +\infty$ . If there is a  $\delta > 0$ , such that for every  $R > 0, \varepsilon > 0$ , there is a map  $\varphi : X \rightarrow E$  satisfying:*

- (1)  $\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \leq R\} \leq \varepsilon$ .
  - (2)  $\forall m \in \mathbb{N}, \sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \leq m\} < +\infty$ .
  - (3)  $\lim_{s \rightarrow +\infty} \inf\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \geq s\} \geq \delta$ .
- then  $X$  admits a coarse embedding into  $E^p$ .*

This is generalized from the conditions for coarse embedding into a Hilbert space[3]. Using this condition, we study the coarse embedding under gluing property. This is easy to prove in the case of Hilbert space, but difficult in the case of Banach space. We only obtained some partial results.

**Proposition 1.2.** *Let  $X$  be a metric space, and  $X = X_1 \cup X_2$ , with  $X_1, X_2$  admit coarse embedding into a Banach space  $E$  and  $1 \leq p < +\infty$ . If for any  $s > 0$ , there is a bounded set  $C_s$  such that the sets  $\{X_i \setminus C_s\}$  is  $s$ -separated, then  $X$  admits a coarse embedding into  $E^p$ .*

Recall that two subsets  $X_1, X_2$  of a metric space  $X$  are  $s$ -separated if  $d(X_1, X_2) = \inf\{d(x, y), x \in X_1, y \in X_2\} \geq s$ .

**Proposition 1.3.** *If  $X$  is long range disconnected at infinity and all  $\{X_i^n\}$  are equivalently coarse embedded into a Banach space  $E$  by coarse maps  $\{\varphi_i^n\}$ , then  $X$  admits a coarse embedding into  $E^p$ .*

In case of infinite union, we have

**Proposition 1.4.** *Let  $X$  be a metric space with  $X = \cup_{i \in I} X_i$ , if for any  $s > 0$ , there is a bounded set  $C_s$  with  $X_i \cap C_s \neq \emptyset$  for any  $i$  and the sets  $\{X_i \setminus C_s\}$  is  $s$ -separated. If  $\{X_i\}$  can be equally coarse embedded into  $E$ , then  $X$  can be coarse embedded into  $E^p$ .*

Further, we study the coarse embeddability of the relative hyperbolic group and prove that:

**Theorem 1.5.** *If the group  $G$  is strongly relative hyperbolic to a subgroup  $H$  and  $H$  admits a coarse embedding into a uniformly convex Banach space  $E$ , let  $B(n) = \{g \in G \mid |g|_{S \cup \mathcal{H}} \leq n\}$ , then  $B(n)$  admits a coarse embedding into  $E^p$ .*

The author would like to show his thanks to Professor X.Chen and Professor G.Yu for helpful discussion.

## 2. Coarse geometry and convex Banach space

We first recall some definitions in coarse geometry[13].

**Definition 2.1.** Let  $X, Y$  be metric spaces, and  $f$  be a map from  $X$  to  $Y$ :

- (1) The map  $f$  is *proper* if the inverse image, under  $f$ , of any bounded subset of  $Y$ , is a bounded subset of  $X$ .
- (2) The map is *bornologous* if for every  $R > 0$  there is an  $S > 0$ , such that  $d(x, y) \leq R$  implies  $d(f(x), f(y)) \leq S$ .
- (3)  $f$  is *coarse* if it is proper and bornologous.

And we say that  $X$  admits a coarse embedding into  $Y$  if there is a coarse map  $f : X \rightarrow Y$ . We usually consider the case where  $Y$  is a Banach space. We will often need to deal with a family of metric spaces.

**Definition 2.2.** A family of metric space  $\{X_i\}_{i \in I}$  is called equivalently coarse embedded into a metric space  $Y$  if there exist  $\{f_i : X_i \rightarrow Y\}_{i \in I}$  satisfying:

- (1)  $\forall s \geq 0$ , there exist  $S \geq 0$ , if  $d(x_i, x'_i) \leq s$  then  $d(f_i(x_i), f_i(x'_i)) \leq S$  for all  $i \in I$ .
- (2)  $\forall r \geq 0$ , there exist  $R \geq 0$ , if  $d(f_i(x_i), f_i(x'_i)) \leq r$  then  $d(x_i, x'_i) \leq R$  for all  $i \in I$ .

Uniformly convex Banach space is an important object to study in classical Banach space theory[9].

**Definition 2.3.** A Banach space  $E$  is called uniformly convex if for any  $\epsilon > 0$ , there exists a  $\delta > 0$ , for any  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \epsilon$ , then  $\left\| \frac{x+y}{2} \right\| \leq 1 - \delta$ .

We know  $\ell^p(1 < p < +\infty)$  is uniformly convex Banach space. If  $E$  is a uniformly convex Banach space, let

$$E^p = \{x = (x_i)_{i \in \mathbb{N}} \mid x_i \in E(i \in \mathbb{N}), \sum_{n \in \mathbb{N}} \|x_i\|^p < +\infty\}$$

with the norm  $\|x\| = (\sum_{n \in \mathbb{N}} \|x_n\|^p)^{\frac{1}{p}}$ . If  $1 < p < +\infty$ ,  $E^p$  is also a uniformly convex Banach space.

### 3. Coarse embedding into uniformly convex Banach space

We first rewrite the condition for coarse embedding into a uniformly convex Banach space.

**Theorem 3.1.** *Let  $X$  be a metric space and  $E$  be a Banach space, and  $1 \leq p < +\infty$ . If there is a  $\delta > 0$  such that for every  $R > 0, \varepsilon > 0$  there is a map  $\varphi : X \rightarrow E$  satisfying:*

- (1)  $\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \leq R\} \leq \varepsilon$ .
- (2)  $\forall m \in \mathbb{N}, \sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \leq m\} < +\infty$ .
- (3)  $\lim_{s \rightarrow +\infty} \inf\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \geq s\} \geq \delta$ .

*then  $X$  admits a coarse embedding into  $E^p$ .*

*Proof.* For  $n \in \mathbb{N}$ , let  $R_n = n$ ,  $\varepsilon = \frac{1}{2^n}$ , there is a  $\varphi_n : X \rightarrow E$  satisfying the above conditions. And we can find an  $s_n$  such that  $\|\varphi_n(x) - \varphi_n(y)\| > \frac{\delta}{2}$  if  $d(x, y) \geq s_n$ , we can choose  $\{s_n\}$  to be an increasing sequence. Fix a point  $x_0 \in X$  and define

$$\begin{aligned} \varphi : X &\rightarrow E^p \\ x &\mapsto \bigoplus_{n=1}^{\infty} (\varphi_n(x) - \varphi_n(x_0)) \end{aligned}$$

it is easy to see that  $\|\varphi(x)\| < +\infty$  for each  $x \in X$ . We show that  $\varphi$  is coarse.

(1) For any  $x, y \in X$ , assume  $k-1 < d(x, y) \leq k$ , then

$$\begin{aligned} \|\varphi(x) - \varphi(y)\|^p &= \sum_{n=1}^{+\infty} \|\varphi_n(x) - \varphi_n(y)\|^p \\ &= \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|^p + \sum_{n=k}^{+\infty} \|\varphi_n(x) - \varphi_n(y)\|^p \\ &\leq \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|^p + \sum_{n=k}^{+\infty} \frac{1}{2^{np}}. \end{aligned}$$

Let  $C_n^k = \sup\{\|\varphi_n(x) - \varphi_n(y)\|, d(x, y) \leq k\}$ , then

$$\|\varphi(x) - \varphi(y)\|^p \leq \sum_{n=1}^{k-1} (C_n^k)^p + 1.$$

(2) For any  $x, y \in X$ , assume  $s_{k-1} \leq d(x, y) \leq s_k$ , then

$$\|\varphi(x) - \varphi(y)\|^p = \sum_{n=1}^{\infty} \|\varphi_n(x) - \varphi_n(y)\|^p \geq \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|^p > (k-1)\left(\frac{\delta}{2}\right)^p$$

and  $d(x, y) \rightarrow +\infty$  implies  $k \rightarrow +\infty$ , so  $(k-1)\left(\frac{\delta}{2}\right)^p \rightarrow +\infty$ .  $\square$

**Example 3.2.**  $\ell^p$  ( $1 \leq p < +\infty$ ) satisfies the above conditions for  $E = \ell^p$ .

*Proof.* Let  $\delta = 1$ ,  $\forall R > 0, \varepsilon > 0$ , there is a natural number  $h$  such that  $\frac{R}{h} < \varepsilon$ , define  $\varphi : \ell^p \rightarrow \ell^p$  by  $\varphi(x) = \frac{x}{h}$  then

(1)  $\sup\{\|\varphi(x) - \varphi(y)\|, d(x, y) \leq R\} = \sup\{\frac{1}{h}\|x - y\|, d(x, y) \leq R\} < \varepsilon$ .

(2)  $\sup\{\|\varphi(x) - \varphi(y)\|, d(x, y) \leq m\} = \frac{m}{h} < +\infty$ .

(3)  $\inf\{\|\varphi(x) - \varphi(y)\|, d(x, y) \geq s\} = \frac{s}{h}$ ,  $\lim_{s \rightarrow +\infty} \frac{s}{h} = +\infty$ .  $\square$

W.B.Johnson and N.L.Randrianarivony proved that  $\ell^p$  ( $p > 2$ ) does not admit a coarse embedding into a Hilbert space[8]. So the conditions for coarse embedding into a uniformly convex Banach space  $E$  in the above theorem is very different from the coarse embedding into Hilbert space[3].

**Example 3.3.** If  $X$  admits a coarse embedding into  $\ell^p$ , then  $X$  satisfies the above conditions.

*Proof.* There is a  $\psi : X \rightarrow \ell^p$  and  $\rho_{\pm}$ , such that

(1)  $\rho_-(d(x, y)) \leq \|\psi(x) - \psi(y)\|_p \leq \rho_+(d(x, y))$ ,

(2)  $\lim_{r \rightarrow +\infty} \rho_+(r) = +\infty$ .

$\forall R > 0, \varepsilon > 0$ , there is a nature number  $h$  such that  $\frac{\rho_+(R)}{h} < \varepsilon$

define  $\varphi : X \rightarrow \ell^p$  by  $\varphi(x) = \frac{\psi(x)}{h}$  then:

(1)  $\sup\{\|\varphi(x) - \varphi(y)\|_p, d(x, y) \leq R\} = \sup\{\frac{\|\psi(x) - \psi(y)\|_p}{h}, d(x, y) \leq R\} \leq \varepsilon$ .

(2)  $\sup\{\|\varphi(x) - \varphi(y)\|_p, d(x, y) \leq m\} = \sup\{\frac{\|\psi(x) - \psi(y)\|_p}{h}, d(x, y) \leq m\} \leq \frac{\rho_+(m)}{h}$ .

(3)  $\inf\{\|\varphi(x) - \varphi(y)\|_p, d(x, y) \geq s\} = \inf\{\frac{\|\psi(x) - \psi(y)\|_p}{h}, d(x, y) \geq s\} \geq \frac{\rho_-(s)}{h}$

and  $\lim_{s \rightarrow +\infty} \frac{\rho_-(s)}{h} = +\infty$ .  $\square$

*Remark 3.4.* (a) We can see from the proof that this  $\delta$  is not important. We can take it to be infinity in general, i.e, replace third condition with

$$\lim_{s \rightarrow +\infty} \inf \{ \|\varphi(x) - \varphi(y)\|_p : x, y \in X, d(x, y) \geq s \} = +\infty.$$

(b) If we take  $E = \ell^p(1 < p < +\infty)$  and change the condition (2) of Theorem 3.1 with

$$\sup \{ \|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \leq m \} < L_0, \quad \forall m \in \mathbb{N}$$

for some fixed  $L_0$ , by the Mazur map mentioned in [1], then it can be embedded into  $\ell^2$ , a Hilbert space.

#### 4. On the union of metric spaces

In this section, we study the coarse embeddability under taking the union of metric spaces.

**Proposition 4.1.** *Let  $X$  be a metric space, and  $X = X_1 \cup X_2$ , with  $X_1, X_2$  admit coarse embedding into a Banach space  $E$  and  $1 \leq p < +\infty$ . If for any  $s > 0$ , there is a bounded set  $C_s$  such that the sets  $\{X_i \setminus C_s\}$  is  $s$ -separated, then  $X$  admits a coarse embedding into  $E^p$ .*

*Proof.* We first assume that  $X_1 \cap X_2 \neq \emptyset$ , take an  $x_0 \in X_1 \cap X_2$ , and replace  $C_s$  with  $C_s \cup x_0$  if necessary, we can assume that  $x_0 \in C_s$  for any  $s$ .

For any  $R \geq 0$ ,  $\varepsilon \geq 0$ , there is a bounded set  $C_{2R}$  such that  $X_i \setminus C_{2R}$  is  $2R$ -separated. Suppose  $C_{2R} \subset B(x_0, k)$  for some  $k$ . For  $X_i$  admits coarse embedding into  $E$ , we can find a number  $r > 2R + k$  and a map  $\varphi_r^i$ , such that

- (1)  $\sup \{ \|\varphi_r^i(x) - \varphi_r^i(y)\| : x, y \in M_i, d(x, y) \leq r \} \leq \varepsilon$ .
- (2)  $\forall m \in \mathbb{N}, \sup \{ \|\varphi_r^i(x) - \varphi_r^i(y)\| : x, y \in M_i, d(x, y) \leq m \} < +\infty$ .
- (3)  $\lim_{s \rightarrow +\infty} \inf \{ \|\varphi_r^i(x) - \varphi_r^i(y)\| : x, y \in M_i, d(x, y) \geq s \} = +\infty$ .

And we define

$$\begin{aligned} \varphi : \quad X &\rightarrow (E \oplus E)_p \\ x &\mapsto (\varphi_r^1(a) - \varphi_r^1(x_0), 0) \text{ if } x \in M_1 \setminus C_{2R}, \\ y &\mapsto (0, \varphi_r^2(b) - \varphi_r^2(x_0)) \text{ if } y \in M_2 \setminus C_{2R}, \\ z &\mapsto (0, 0) \text{ if } z \in C_{2R} \end{aligned}$$

We need to verify the conditions in Theorem 3.1:

(i) For  $d(x, y) \leq R$ , if  $x \in M_1 \setminus C_{2R}, y \in C_{2R}$ , then  $d(x_0, y) \leq k$ , we have  $d(x, x_0) \leq d(x, y) + d(y, x_0) \leq k + R$ , so

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^1(x) - \varphi_r^1(x_0)\| < \epsilon.$$

If  $x \in M_2 \setminus C_{2R}, y \in C_{2R}$  or  $x, y \in C_{2R}$  or  $x, y \in M_i \setminus C_{2R}$  for same  $i$ , it is similar to prove  $\|\varphi(x) - \varphi(y)\| < \epsilon$ .

(ii)  $\forall m > 0$ , there is a bounded set  $C_m$ , such that  $X_i \setminus C_m$  is  $m$ -separated, and we can find a number  $h$  such that  $C_m \subset B(C_{2R}, h)$ . For  $d(x, y) < m$ , if  $x \in X_i \setminus C_{2R}, y \in X_i \setminus C_{2R}$  for same  $i$  then

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^i(x) - \varphi_r^i(y)\|.$$

if  $x \in X_i \setminus C_{2R}, y \in C_{2R}$ , then  $d(x, x_0) \leq d(x, y) + d(y, x_0) \leq m + k$ . And

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^i(x) - \varphi_r^i(x_0)\|.$$

if  $x \in X_1 \setminus C_{2R}, y \in M_2 \setminus C_{2R}$ , for  $d(x, y) \leq m$ , then either  $x \in C_m$  or  $y \in C_m$ . Suppose  $x \in C_m$ , so  $d(x, x_0) \leq h + k; d(y, x_0) \leq h + k + m$ , then

$$\|\varphi(x) - \varphi(y)\| = (\|\varphi_r^1(x) - \varphi_r^1(x_0)\|^p + \|\varphi_r^2(y) - \varphi_r^2(x_0)\|^p)^{\frac{1}{p}}.$$

Let  $t = h + m + k$ , we get

$$\begin{aligned} \sup\{\|\varphi(x) - \varphi(y)\|, d(x, y) \leq m\} &\leq \max\left\{\sup_{d(x, y) \leq t} \|\varphi_r^i(x) - \varphi_r^i(y)\|, \right. \\ &\left. \sup_{d(x, x_0) \leq t, d(y, x_0) \leq t} \{(\|\varphi_r^1(x) - \varphi_r^1(x_0)\|^p + \|\varphi_r^2(y) - \varphi_r^2(x_0)\|^p)^{\frac{1}{p}}\} \right\} < +\infty. \end{aligned}$$

(iii) Let  $d(x, y) = s$ , let  $s$  tends to infinity,

if  $x, y \in X_i \setminus C_{2R}$ , then  $\|\varphi_r^i(x) - \varphi_r^i(y)\| \rightarrow +\infty$  by the property of  $\varphi_r^i$ .

if  $x \in X_i \setminus C_{2R}, y \in C_{2R}$ , then for  $d(y, x_0) < k$ , so  $d(x, y) \rightarrow +\infty$  implies  $d(x, x_0) \rightarrow +\infty$ , so

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^i(x) - \varphi_r^i(x_0)\| \rightarrow +\infty.$$

if  $x \in X_1 \setminus C_{2R}, y \in X_2 \setminus C_{2R}$ ,  $d(x, y) \rightarrow +\infty$ , implies either  $d(x, x_0) \rightarrow +\infty$  or  $d(y, x_0) \rightarrow +\infty$ , thus

$$\|\varphi(x) - \varphi(y)\|^p = (\|\varphi_r^1(x) - \varphi_r^1(x_0)\|^p + \|\varphi_r^2(y) - \varphi_r^2(x_0)\|^p)^{\frac{1}{p}} \rightarrow +\infty.$$



if  $X_1 \cap X_2 = \emptyset$ , we can assume that  $X_i \cap C_s \neq \emptyset (\forall s > 0)$ , take  $x_0 \in X_1 \cap C_{2R}$ ,  $y_0 \in X_2 \cap C_{2R}$ , and define

$$\begin{aligned}\varphi : X &\rightarrow E \oplus E \\ x &\mapsto (\varphi_r^1(a) - \varphi_r^1(x_0), 0) \text{ if } x \in X_1 \setminus C_{2R}, \\ y &\mapsto (0, \varphi_r^2(b) - \varphi_r^2(y_0)) \text{ if } y \in X_2 \setminus C_{2R}, \\ z &\mapsto (0, 0) \text{ if } z \in C_{2R}\end{aligned}$$

The proof follows. Applying the theorem 3.1, we finish the proof.  $\square$

Gromov introduces the following property for metric space[7].

**Definition 4.2.** A metric space  $X$  is called long range disconnected at infinity if for every  $n \in \mathbb{N}$ , there exist two subsets  $X_1^n$  and  $X_2^n$  in  $X$  such that:  
(1)  $d(X_1^n, X_2^n) = \inf\{d(x_1, x_2) | x_1 \in X_1^n, x_2 \in X_2^n\} \geq d$ .  
(2)  $X_1^n$  and  $X_2^n$  cover almost all  $X$ , i.e.,  $X \setminus (X_1^n \cup X_2^n)$  is bounded.

**Proposition 4.3.** *If  $X$  is long range disconnected at infinity and all  $\{X_i^n\}$  are equivalently coarse embedded into a Banach space  $E$  by coarse maps  $\{\varphi_i^n\}$ , then  $X$  admits a coarse embedding into  $E^p$ .*

*Proof.* For any  $R \geq 0, \epsilon \geq 0$ , find an  $n \in \mathbb{N}$  such that  $n > R$ . Choose a point  $x_n \in X \setminus (X_1^n \cup X_2^n)$ , define

$$\begin{aligned}\varphi : X &\rightarrow (E \oplus E)_p \\ x &\mapsto (\varphi_1^n(a) - \varphi_1^n(x_n), 0) \text{ if } x \in X_1^n, \\ y &\mapsto (0, \varphi_2^n(b) - \varphi_2^n(x_n)) \text{ if } y \in X_2^n, \\ z &\mapsto (0, 0) \text{ otherwise.}\end{aligned}$$

Using the similar argument in proposition 4.1, it can be show that  $\varphi$  satisfies the condition of theorem 3.1, we finish the proof.  $\square$

**Proposition 4.4.** *Let  $X = \cup_{i \in I} X_i$  be a metric space. If for any  $s > 0$ , there is a bounded set  $C_s$  with  $X_i \cap C_s \neq \emptyset (\forall i)$  and the sets  $\{X_i \setminus C_s\}$  is  $s$ -separated. If  $X_i$  can be equally coarse embedded into a Banach space  $E$ , then  $X$  can be coarse embedded into  $E^p$ .*

*Proof.*  $\forall R > 0, \epsilon > 0$ , there is a bounded set  $C_R$  such that  $M_i \setminus C_R$  is R-separated, and suppose that  $C_R \subset B(x_0, k)$  for some  $k$ , take an  $r > k + 2R$ . For  $X_i$  is equally coarse embedded into  $E$ , we can find  $\varphi_r^i : X_i \rightarrow E$ , such that

- (1)  $\sup_i \sup\{\|\varphi_r^i(x) - \varphi_r^i(y)\| < \epsilon, x, y \in X_i, d(x, y) < r\} < \epsilon$ .
- (2)  $\sup_i \sup\{\|\varphi_r^i(x) - \varphi_r^i(y)\|, x, y \in X_i, d(x, y) < m\} < \infty, \forall m \in \mathbb{N}$ .
- (3)  $\lim_{s \rightarrow \infty} \inf_i \inf\{\|\varphi_r^i(x) - \varphi_r^i(y)\|, x, y \in X_i, d(x, y) > s\} = \infty$ .

For each  $i$ , fix an  $x_i \in X_i \cap C_R$ . And define

$$\begin{aligned} \varphi : \quad & M \rightarrow E^p \\ & a \mapsto (0, \dots, \underset{\text{ith item}}{\varphi_r^i(x) - \varphi_r^i(x_i)}, 0, \dots) \text{ if } a \in M_i \setminus C_R \\ & b \mapsto (0, \dots, 0) \text{ if } b \in C_R \end{aligned}$$

The proof follows using the similar argument in proposition 4.1.  $\square$

## 5. Relative hyperbolic group

Let  $G$  be a finitely generated group with generating set  $S$  (closed under taking inverse) then  $G$  is a proper metric space with word length metric induced by the generating set  $S$ . Let  $H$  be a finite generated subgroup of  $G$ . We denoted  $\{H \setminus e\}$  by  $\mathcal{H}$ . Then the Cayley graph  $(G, S)$  and  $(G, S \cup \mathcal{H})$  are both metric spaces with word length metric  $d_S, d_{S \cup \mathcal{H}}$ , respectively.

**Definition 5.1.** Let  $p$  be a path in  $(G, S \cup \mathcal{H})$ . An  $\mathcal{H}$ -component of  $p$  is a maximal sub-path of  $p$  contained in a same left coset  $gH$ . The path is said to be *without backtracking* if it does not have two distinct  $\mathcal{H}$ -component in a same coset  $gH_i$ .

**Definition 5.2.** a path-metric space  $X$  is hyperbolic if there exists some  $\delta > 0$  such that the  $\delta$ -neighborhood of any two sides of a geodesic triangle contain the third side. The group  $G$  is said to be *weakly hyperbolic relative to  $H$*  if the Cayley graph  $(G, S \cup \mathcal{H})$  is hyperbolic.

**Definition 5.3 (see[5]).** We say the pair  $(G, H)$  satisfies the *Bounded Coset Penetration property (BCP)* if for every  $R \geq 0$ , there exists  $a = a(R)$  such that if  $p, q$  are two geodesics in  $(G, S \cup \mathcal{H})$  with  $p_- = q_-$  and  $d_S(p_+, q_+) \leq R$ , (1) Suppose that  $p$  has an  $\mathcal{H}$ -component  $s$  with  $d_S(s_-, s_+) \geq a(R)$ , then  $q$

has an  $\mathcal{H}$ -component contains in the same left coset of  $s$ .

(2) Suppose  $s, t$  are two  $\mathcal{H}$ -component of  $p, q$  respectively, contained in the same left coset, then  $d_S(s_-, t_-) \leq a(R)$ ,  $d_S(s_+, t_+) \leq a(R)$ .

**Definition 5.4.** The group  $G$  is *strongly relative hyperbolic* to  $H$  if it is weakly hyperbolic to  $H$  and satisfies *BCP*.

Denote by  $B(n) = \{g \in G \mid |g|_{S \cup \mathcal{H}} \leq n\}$ . D.Osin proved in [12] that  $B(n)$  has asymptotic dimension at most  $d$  if the subgroup  $H$  have asymptotic dimension at most  $d$ . M.Dadarlat and E.Guentner proved  $G$  admits a coarse embedding into a Hilbert space if  $H$  admits a coarse embedding into a Hilbert space [4]. We prove that:

**Theorem 5.5.** *If  $H$  admits a coarse embedding into  $E$ , then  $B(n)$  admits a coarse embedding into  $E^p$  for each  $n \in \mathbb{N}$ .*

*Proof.* we proceed by induction on  $n$ . We have  $B(0) = \{e\}$  is trivial.  $B(1) = H \cup S$  is just in the 1-neighborhood of  $H$ , so can be coarsely embedded. We assume that  $B(n-1)$  is coarsely embedded into  $\ell^p$ . we know

$$B(n) = \left( \bigcup_{x \in S} B(n-1)x \right) \cup B(n-1)H$$

since  $B(n-1)x$  is just in the 1-neighborhood of  $B(n-1)$  in  $(G, S)$ , so it can be coarsely embedded. We concerned on  $B(n-1)H$ . we can find a subset  $R(n-1)$  in  $B(n-1)$  such that for any  $b \in B(n-1)$ ,  $bH = gH$  for a unique  $g \in R(n-1)$ . Thus

$$B(n-1)H = \bigsqcup_{g \in R(n-1)} gH.$$

$\forall R \geq 0$ ,  $\epsilon \geq 0$ , we have an  $a(R)$  from the *BCP*. We can assume  $a(R) \geq R$  and  $a(R)$  is increasing. let  $T_R = \{g \in G \mid |g|_S \leq a(R)\}$ . Let  $Y_R = B(n-1)T_R$ , then D.Osin proved that  $\{gH \setminus Y_R\}_{g \in R(n-1)}$  is  $R$ -separated [12]. We find maps  $\varphi_1$  and  $\varphi_2$  for embedding of  $Y_R$  and  $H$ , respectively, such that

- (1)  $\sup\{\|\varphi_i(x) - \varphi_i(y)\|_p : x, y \in X, d_S(x, y) \leq 3a(R)\} \leq \epsilon/2$ ,
- (2)  $C_m = \sup\{\|\varphi_i(x) - \varphi_i(y)\|_p : x, y \in X, d_S(x, y) \leq m\} < +\infty \quad \forall m \in \mathbb{N}$ ,
- (3)  $\lim_{t \rightarrow +\infty} \inf\{\|\varphi_i(x) - \varphi_i(y)\|_p : x, y \in X, d_S(x, y) \geq t\} \geq \delta$ .

We define a map:

$$\varphi : B(n-1)H \rightarrow E \oplus \left( \bigoplus_{g_i \in R(n-1)} E \right)$$

as following.

For  $x \in g_i H \setminus Y_R$ , fix a shortest word  $A_i$  for  $g_i$  in  $(G, S \cup \mathcal{H})$ . Let  $A_i = g'_i h'_i$  where  $h'_i$  is the  $\mathcal{H}$ -component in  $g_i H$ . Replacing  $g_i$  with  $g'_i$ , we can assume  $g_i$  does not have an  $\mathcal{H}$ -component in  $g_i H$ . Then  $x = g_i x_i$  is a geodesic in  $(G, S \cup \mathcal{H})$  [12]. We define  $\varphi(x) = \varphi_1(g_i) \oplus \varphi_2(x_i)$  where  $\varphi_2(x_i)$  is in the  $g_i$  item in  $\bigoplus_{g_i \in R(n-1)} E$ . And let  $\varphi(y) = \varphi_1(y)$  for  $y \in Y_R$ . We need to verify the three conditions of embedding.

(1) For  $d_S(x, y) \leq R$ , we have two cases.

Case a: If  $x, y \in Y_R$ ,  $\|\varphi_1(x) - \varphi_1(y)\| \leq \epsilon$ .

Case b: If  $x \in g_i H \setminus Y_R$ , we have  $y \in g_i H$  by *BCP*. If  $y \in g_i H \setminus Y_R$ , let  $y = g_i y_i$ ,  $x = g_i x_i$  be the geodesics in  $(G, S \cup \mathcal{H})$ . Thus  $d_S(x_i, y_i) = d_S(x, y) \leq R$ ,

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_2(x_i) - \varphi_2(y_i)\| < \epsilon.$$

If  $y \in g_i H \cap Y_R$ , let  $y = g'_i y'_i$  be a geodesic in  $(G, S \cup \mathcal{H})$ . For  $d_S(x, y) \leq R$ , we have  $d_S(g_i, g'_i) \leq a(R)$  and  $|y_i|_s \leq a(R)$ . Then

$$d_S(g_i, y) \leq d_S(g_i, g'_i) + d_S(g'_i, y) \leq 2a(R),$$

$$|x_i|_s = d_S(g_i, x) \leq d_S(g_i, y) + d_S(y, x) \leq 3a(R).$$

we have  $\|\varphi(x) - \varphi(y)\| = (\|\varphi(g_i) - \varphi(y)\|^p + \|\varphi_2(x_i)\|^p)^{\frac{1}{p}} \leq \epsilon$ .

(2) For  $m \in \mathbb{N}$  and  $d_S(x, y) \leq m$ , let  $T_m = \{g \mid |g|_S \leq a(m)\}$  and  $Y_m = B(n-1)T_m$ . Then  $\{g_i H \setminus Y_m\}$  is  $m$ -separated. If  $x, y$  is both in  $Y_R$  or in  $g_i H$  for some  $i$ , it is easy to see  $\|\varphi(x) - \varphi(y)\|$  is bounded. So we only need to consider  $x \in g_i H$ ,  $y \in g_j H$  with  $i \neq j$ . For  $d_S(x, y) \leq m$ , either  $x$  or  $y$  is in  $Y_m$ . We assume  $y \in Y_m$ . For  $x \in g_i H \cap (Y_m \setminus Y_R)$ , let  $x = g_i x_i$  be a geodesic

in  $(G, S \cup \mathcal{H})$ . We have two cases,

Case a: If  $y \in g_j H \cap (Y_m \setminus Y_R)$ . let  $y = g_j y_j$  be geodesics in  $(G, S \cup \mathcal{H})$ . Then

$$\|\varphi(x) - \varphi(y)\|^p = \|\varphi_1(g_i) - \varphi_1(g_j)\|^p + \|\varphi_2(x_i)\|^p + \|\varphi_2(y_j)\|^p.$$

for  $d_S(g_i, g_j) \leq a(m)$ ,  $|x|_S \leq a(m)$ ,  $|y_j|_S \leq a(m)$ , we know  $\|\varphi(x) - \varphi(y)\| \leq 3C_m$ .

Case b: If  $y \in g_j H \cap Y_R$ , let  $y = g'_i y'_i$  be the geodesic in  $(G, S \cup \mathcal{H})$  with  $y_i$  is a  $\mathcal{H}$ -component and  $|y'_i|_S \leq a(R)$ . For  $d_S(x, y)$ , then  $d_S(g_i, g'_i) \leq a(m)$  and  $d_S(g_i, y) \leq 2a(m)$ ,  $|x|_S \leq 3a(m)$ . Then

$$\|\varphi(x) - \varphi(y)\|^p = \|\varphi_1(g_i) - \varphi_1(y)\|^p + \|\varphi_2(x_i)\|^p.$$

we have that  $\|\varphi(x) - \varphi(y)\| \leq 2C_m$ .

(3) We have  $d_S(x, y) \leq l(x_g) + d_S(g_i, g_j) + l(y_g)$ , thus  $d_S(x, y)$  tends infinity implies at least one the three must tends to infinity. So

$$\lim_{t \rightarrow +\infty} \inf \{ \|\varphi(x) - \varphi(y)\|, d_S(x, y) \geq t \} = \infty.$$

By Theorem 3.1,  $B(n)$  admits a coarse embedding into  $E^p$ . □

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